International Journal of Theoretical Physics, Vol. 30, No. 12, 1991

Generalized Mass-Scattering Integrals for Dirac Fields and Their Graphical Representation

J. G. Cardoso[1]

Received December 27, 1990

Penrose suggested that Dirac fields could be constructed from an infinite number of elementary distributional fields scattering off each other, with the mass of the entire fields playing the role of a coupling constant. Following this suggestion, we present a complete null description of the mass-scattering processes. The general pattern of the null initial data for successive processes is explicitly exhibited. The entire fields are given by four series of terms, each being a manifestly finite scaling-invariant integral which is taken over a compact space of appropriate mass-scattering zigzags. A set of simple rules which enable one to evaluate any term of the series in a graphical way is given. These rules give rise to a colored-graph representation of the scattering integrals.

1. INTRODUCTION

Penrose's null initial data (NID) techniques constitute a powerful instrument for describing the dynamics of sets of interacting spinor fields in real Minkowski space $\mathbb{RM}$ (Penrose, 1980; Penrose and Rindler, 1984). In this framework, the key concept is that of an invariant exact set, which guarantees a consistent description of the field dynamics. Likewise, the initial data for all the relevant processes are specified at nonsingular points of null hypersurfaces in $\mathbb{RM}$. The evaluation of the entire fields is then carried out by using either integral devices or appropriate power series expansions.

Penrose suggested (Penrose and Rindler, 1984) that, regarding the Dirac fields as the elements of an invariant exact set of interacting spinor fields on $\mathbb{RM}$, one could use these techniques to build up a solution of the source-free Dirac equation. In accordance with this procedure, the Van der Waarden form of the field equation should be employed such that the mass of the entire fields would appear as a coupling constant. The entire Dirac fields

[1]Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

0020-7748/91/1200-1565$06.50/0

would then be given as a linear superposition involving an infinite number of elementary distributional fields, which propagate for a while along null geodesics of $\mathbb{RM}$, and repeatedly scatter off each other at points lying in the interior of the future cone of an origin of $\mathbb{RM}$.

In this paper, we develop Penrose's suggestion further to describe completely the mass scattering of Dirac fields in $\mathbb{RM}$. All the densities associated with the elementary contributions satisfy Dirac-like (proper Lorentz-invariant) distributional field density equations on the interior of the future cone of the origin (Section 2). The initial data generating the elementary fields are specified at points of the future null cone of the origin. Thus, the successive mass-scattering processes take place at suitable vertices of appropriate null zigzags that start at the origin and terminate at a fixed point of the interior of the future cone of the origin. These vertices are indeed the points at which the NID for the scattering processes are explicitly evaluated. Each element of the infinite Dirac set appears then to be given as a scaling-invariant (SI) Kirchhoff-like integral (Cardoso, 1990*a*) which is taken over a suitable space of such null graphs (Section 3). Hence, as far as the scattering processes are concerned, the propagation of all elementary fields is viewed as taking place entirely along the edges of these graphs. A representation of the field integrals in terms of colored graphs associated with mass-scattering zigzags leads us to four graphical expansions for the entire fields (Section 4).

Throughout this work all the fields are looked upon as classical, and there will be no attempt at this stage to regard them as quantized fields. We shall make use of some basic properties of distributional fields on $\mathbb{RM}$ (Penrose and Rindler, 1984).

2. FIELD DENSITY EQUATIONS

The main aim of this section is to present the distributional field density equations (DFDE). We endow the null graphs, which appear to play an important role here, with appropriate sets of spin bases. These bases enter into the definition of the relevant NID sets.

2.1. Forward Mass-Scattering Null Zigzags

Let V_0^+ denote the interior of the future cone of an origin O of $\mathbb{RM}$. Let, also, $\mathscr{C}_0^+$ be the future null cone of O. A forward null geodesic zigzag ζ_N is a simple graph (see, for example, Busacker and Saaty, 1965) whose vertex-set is the set

$$V(\zeta_N)=\{\overset{0}{x}{}^{AA'}, \overset{1}{x}{}^{AA'}, \overset{2}{x}{}^{AA'}, \ldots, \overset{N}{x}{}^{AA'}\} \tag{2.1}$$

of $N+1$ vertices all belonging to $\mathbb{RM}$, with $N \geq 2$. The vertex $\overset{n+1}{x}{}^{AA'}$ lies on the (future) forward null cone C_n^+ of $\overset{n}{x}{}^{AA'}$, $n=0, 1, 2, \ldots, N-1$. The edge-set of ζ_N is the set

$$E(\zeta_N) = \{\overset{1}{r}, \overset{2}{r}, \overset{3}{r}, \ldots, \overset{N}{r}\} \tag{2.2}$$

of N suitable affine parameters, where each $\overset{n+1}{r}$ lies on the (null geodesic) generator γ_{n+1} of C_n^+ that passes through $\overset{n}{x}{}^{AA'}$ and $\overset{n+1}{x}{}^{AA'}$. The elements $\overset{0}{x}{}^{AA'}$ and $\overset{N}{x}{}^{AA'}$ of (2.1) are the starting-vertex and end-vertex of $V(\zeta_N)$, respectively.

We next introduce the set of N pairs of forward spin bases

$$\begin{aligned}\mathrm{FSBS} = \{\{\{\overset{0}{o}{}^{A}, \overset{1}{o}{}^{A}\}, \{\underset{0}{\bar{o}}{}^{A'}, \underset{1}{\bar{o}}{}^{A'}\}\}, \{\{\overset{1}{o}{}^{A}, \overset{2}{o}{}^{A}\}, \{\underset{1}{\bar{o}}{}^{A'}, \underset{2}{\bar{o}}{}^{A'}\}\},\\ \ldots\{\{\overset{N-1}{o}{}^{A}, \overset{N}{o}{}^{A}\}, \{\underset{N-1}{\bar{o}}{}^{A'}, \underset{N}{\bar{o}}{}^{A'}\}\}\}\end{aligned} \tag{2.3}$$

where

$$\{\{\overset{n}{o}{}^{A}, \overset{n+1}{o}{}^{A}\}, \{\underset{n}{\bar{o}}{}^{A'}, \underset{n+1}{\bar{o}}{}^{A'}\}\}$$

is set up at the vertex $\overset{n}{x}{}^{AA'} \in V(\zeta_N)$, n running over the same values as before. This set is here called a forward spin-basis set. The pair

$$\{\{\overset{0}{o}{}^{A}, \overset{1}{o}{}^{A}\}, \{\underset{0}{\bar{o}}{}^{A'}, \underset{1}{\bar{o}}{}^{A'}\}\}$$

has been introduced even though it does not play any role herein. However, it is relevant when one carries out the twistorial transcription of the mass-scattering formulas (Cardoso, 1988). In (2.3), the spinors $\overset{n+1}{o}{}^{A}$, $\underset{n+1}{\bar{o}}{}^{A'}$ are chosen to be covariantly constant along γ_{n+1}. The real null vectors

$$\{\overset{n}{o}{}^{A}\underset{n}{\bar{o}}{}^{A'}, \overset{n+1}{o}{}^{A}\underset{n+1}{\bar{o}}{}^{A'}\} \tag{2.4}$$

point in (forward) future null directions through $\overset{n}{x}{}^{AA'}$, and the conjugate spin-inner products at $\overset{n}{x}{}^{AA'}$,

$$\overset{n}{z} = \overset{n}{o}{}^{A}\,\overset{n+1}{o}{}_{A}, \qquad \underset{n}{\bar{z}} = \underset{n}{\bar{o}}{}^{A'}\,\underset{n+1}{\bar{o}}{}_{A'} \tag{2.5}$$

are held fixed. Let ζ_N be equipped with an FSBS. In this case, if the starting-vertex of ζ_N is the vertex of $\mathscr{C}_0^+$ and if the elements $\overset{2}{x}$, $\overset{3}{x}{}^{AA'}, \ldots, \overset{N}{x}{}^{AA'}$ of $V(\zeta_N)$ all belong to V_0^+, ζ_N is said to be a mass-scattering null zigzag (MSNZZ).

We now take the element $\overset{N-1}{x}{}^{AA'}$ of the vertex-set of an MSNZZ, and transport the spinors $\overset{N}{o}{}^{A}$, $\underset{N-1}{\bar{o}}{}^{A'}$ (forwardly) parallelly along the generator γ_{N-1} of C_{N-2}^+. An $\overset{N-1}{y}{}^{AA'} \in V_0^+$ that is future-null-separated from an $\overset{N-1}{R}{}^{AA'} \in C_{N-2}^+$ which is, in turn, future-null-separated from $\overset{N-1}{x}{}^{AA'}$ is defined, with respect to $\overset{N-1}{x}{}^{AA'}$, by

$$\overset{N-1}{y}{}^{AA'} = \overset{N-1}{q}\,\overset{N-1}{o}{}^{A}\,\underset{N-1}{\bar{o}}{}^{A'} + \overset{N}{u}\,\overset{N}{o}{}^{A}\,\underset{N}{\bar{o}}{}^{A'} \tag{2.6}$$

In this relation, $\overset{N-1}{q}$ and $\overset{N}{u}$ are appropriate affine parameters, $\overset{N-1}{q}$ lying on γ_{N-1} and $\overset{N}{u}$ lying on the null geodesic that passes through $\overset{N-1}{R}{}^{AA'}$ and $\overset{N-1}{y}{}^{AA'}$ (see Figure 1). Recalling that the inner products $\overset{N-1}{z}$ and $\underset{N-1}{\bar{z}}$ are fixed, and letting $\overset{N-1}{y}{}^{AA'}$ vary suitably, after a short calculation, we obtain

$$\overset{N-1}{\nabla}{}_{AA'}\overset{N}{o}{}^{B} = (-1/\overset{N-1}{z}\,\underset{N-1}{\bar{z}}\,\overset{N}{u})\overset{N}{o}_{A}\,\underset{N-1}{\bar{o}}_{A'}\,\overset{N-1}{o}{}^{B} \tag{2.7}$$

and

$$\overset{N-1}{\nabla}{}_{AA'}\overset{N}{u} = (1/\overset{N-1}{z}\,\underset{N-1}{\bar{z}})\overset{N-1}{o}_{A}\,\underset{N-1}{\bar{o}}_{A'} \tag{2.8}$$

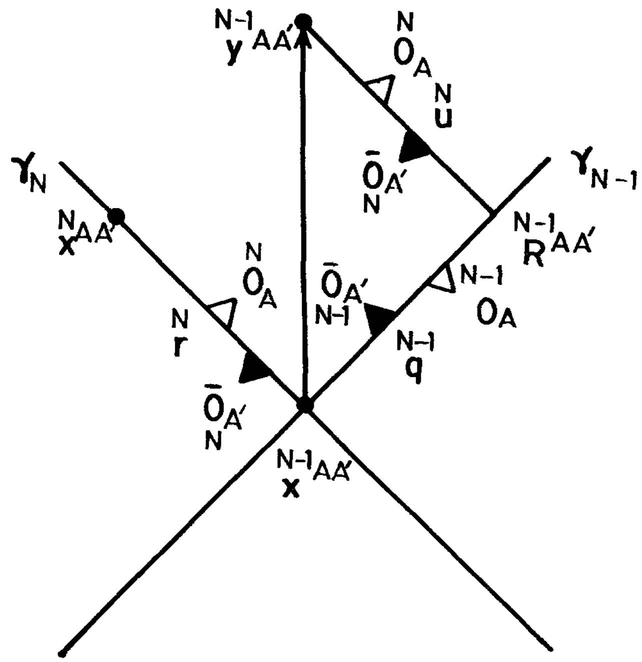


Fig. 1. A point $\overset{N-1}{y}{}^{AA'} \in V_0^+$ is defined with respect to the element $\overset{N-1}{x}{}^{AA'}$ of the vertex-set of an MSZZ.

where $\overset{N-1}{\nabla}{}_{AA'} = \partial/\partial \overset{N-1}{y}{}^{AA'}$. Notice that the spinors $\overset{N-1}{o}{}^{A}$, $\underset{N-1}{\bar{o}}{}^{A'}$ and the vertices $\overset{0}{x}{}^{AA'}, \overset{1}{x}{}^{AA'}, \ldots, \overset{N-1}{x}{}^{AA'}$ have been held fixed at this stage. The complex conjugate of (2.7) is

$$\overset{N-1}{\nabla}{}_{AA'} \underset{N}{\bar{o}}{}^{B'} = (-1/\overset{N-1}{z} \underset{N-1}{\bar{z}} \overset{N}{u}) \overset{N-1}{o}{}_{A} \underset{N}{\bar{o}}{}_{A'} \underset{N-1}{\bar{o}}{}^{B'} \tag{2.9}$$

We assume that an appropriate MSNZZ will be involved in the definition of each of the quantities which are to be considered in what follows. The letters ζ, λ, and η will be sometimes used to refer it to these quantities.

2.2. Infinite Dirac Sets

In the absence of electromagnetism, the two-component spinor form of the Dirac equation on $\mathbb{RM}$ is written as (Penrose and Rindler, 1984)

$$\nabla^{AA'} \psi_A(x) = \mu \chi^{A'}(x), \qquad \nabla_{AA'} \chi^{A'}(x) = -\mu \psi_A(x) \tag{2.10}$$

where $\mu = (2)^{-1/2} \hbar^{-1} m$, m being the mass of the fields and $2\pi\hbar$ the usual Planck constant. Hence the Dirac fields form the invariant exact set of interacting spinor fields (Penrose, 1980)

$$\mathrm{DS} = \{\psi_A(x), \chi^{A'}(x)\} \tag{2.11}$$

the constant μ appearing in (2.10) playing the role of a coupling constant.

The exactness of (2.11) allows us to split it into the infinite invariant exact set on V_0^+

$$\mathrm{IDS} = \{\overset{0}{\psi}_A(x), \underset{0}{\chi}{}^{A'}(x), \overset{1}{\psi}_A(x), \underset{1}{\chi}{}^{A'}(x), \overset{2}{\psi}_A(x), \underset{2}{\chi}{}^{A'}(x), \ldots\} \tag{2.12}$$

whose elements yield a solution of (2.10) whenever we put

$$\psi_A(x) = \sum_{K=0}^{\infty} \overset{K}{\psi}_A(x), \qquad \chi^{A'}(x) = \sum_{K=0}^{\infty} \underset{K}{\chi}{}^{A'}(x) \tag{2.13}$$

In (2.12), $x^{AA'} \in V_0^+$ is effectively identified with the end-vertices

$$\underset{\lambda}{\overset{2K+2}{x}}{}^{AA'}, \quad \underset{\eta}{\overset{2K+3}{x}}{}^{AA'}$$

of MSNZZs λ_{2k+2}, η_{2K+3} such that we can reexpress it as

$$\mathrm{IDS} = \{\overset{0}{\psi}_A(\underset{\lambda}{\overset{2}{x}}), \underset{0}{\chi}{}^{A'}(\underset{\lambda}{\overset{2}{x}}), \overset{1}{\psi}_A(\underset{\eta}{\overset{3}{x}}), \underset{1}{\chi}{}^{A'}(\underset{\eta}{\overset{3}{x}}), \overset{2}{\psi}_A(\underset{\lambda}{\overset{4}{x}}), \underset{2}{\chi}{}^{A'}(\underset{\lambda}{\overset{4}{x}}), \ldots\} \tag{2.14}$$

We thus have

$$\overset{2K}{\psi}_A(x)=\overset{2K}{\psi}_A(\underset{\lambda}{\overset{2K+2}{x}}),\qquad \underset{2K}{\chi}{}^{A'}(x)=\underset{2K}{\chi}{}^{A'}(\underset{\lambda}{\overset{2K+2}{x}}) \tag{2.15a}$$

$$\overset{2K+1}{\psi}{}_A(x)=\overset{2K+1}{\psi}{}_A(\underset{\eta}{\overset{2K+3}{x}}),\qquad \underset{2K+1}{\chi}{}^{A'}(x)=\underset{2K+1}{\chi}{}^{A'}(\underset{\eta}{\overset{2K+3}{x}}) \tag{2.15b}$$

Let ζ_N stand for either λ_{2K+2} or η_{2K+3}. We now associate SI massive distributional field densities with the elements

$$\overset{N-1}{\psi}{}_A(\underset{\zeta}{\overset{N+1}{x}})\quad\text{and}\quad \underset{N-1}{\chi}{}^{A'}(\underset{\zeta}{\overset{N+1}{x}})$$

of (2.14), satisfying the DFDE on V_0^+

$$\underset{\zeta}{\overset{N-1}{\nabla}}{}^{AA'}\overset{N-1}{\Psi}{}_A(\underset{\zeta}{\overset{N-1}{y}})=\mu\underset{N-2}{X}{}^{A'}(\underset{\zeta}{\overset{N-1}{y}}),\qquad \underset{\zeta}{\overset{N-1}{\nabla}}{}_{AA'}\underset{N-1}{X}{}^{A'}(\underset{\zeta}{\overset{N-1}{y}})=-\mu\overset{N-2}{\Psi}{}_A(\underset{\zeta}{\overset{N-1}{y}}) \tag{2.16}$$

The fields

$$\overset{0}{\psi}_A(\underset{\zeta}{\overset{2}{x}})\quad\text{and}\quad \underset{0}{\chi}{}^{A'}(\underset{\zeta}{\overset{2}{x}})$$

are the massless free elements of (2.14) whose SI densities satisfy the DFDE

$$\underset{\zeta}{\overset{1}{\nabla}}{}^{AA'}\overset{0}{\Psi}_A(\underset{\zeta}{\overset{1}{y}})=0,\qquad \underset{\zeta}{\overset{1}{\nabla}}{}_{AA'}\underset{0}{X}{}^{A'}(\underset{\zeta}{\overset{1}{y}})=0 \tag{2.17}$$

The precise meaning of these field densities will be made clear in Section 3. It is evident that the $\underset{\zeta}{y}$'s appearing in (2.16) and (2.17) are of the same form as (2.6).

2.3. Symbolic Expressions for Null Initial Data for Successive Mass-Scattering Processes

The choice of $\mathscr{C}_0^+$ as the NID hypersurface for all the fields in (2.14) enables us to define the NID set (Penrose, 1980; Penrose and Rindler, 1984)

$$\text{NIDS}=\{\overset{0}{\psi}(\overset{1}{o}{}^A;\overset{1}{x}),\ \underset{0}{\chi}(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x}),0,0,\ldots\} \tag{2.18}$$

Its nonvanishing elements are the complex scalar functions on $\mathscr{C}_0^+$

$$\overset{0}{\psi}(\overset{1}{o}{}^A;\overset{1}{x})=\overset{1}{o}{}^A\overset{0}{\psi}_A(\overset{1}{x}),\qquad \underset{0}{\chi}(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x})=\underset{1}{\bar{o}}{}^{A'}\underset{0}{\chi}{}^{B'}(\overset{1}{x})\varepsilon_{B'A'} \tag{2.19}$$

which are specified at the element $\overset{1}{x}{}^{AA'}$ of the vertex-sets of MSNZZs. These NID actually generate the elementary fields of (2.14), and will therefore appear in the explicit expressions for the NID for successive mass-scattering processes (see Section 3.2).

We define the NID hypersurface for the $(N-1)$th mass-scattering process as being the future null cone C^{+}_{N-1} of $\overset{N-1}{x}{}^{AA'} \in V(\zeta_{N+1})$. It is clear that this process involves the mass scattering of the massive field densities

$$\overset{N-1}{\Psi}{}_{A}(\overset{N-1}{y}), \quad \underset{N-1}{X}{}^{A'}(\overset{N-1}{y})$$

The NID set for it is

$$\text{MSNIDS} = \{\overset{N-1}{\psi}(\overset{N}{o}{}^{A}; \overset{N}{x}), \underset{N-1}{\chi}(\underset{N}{\bar{o}}{}^{A'}; \overset{N}{x})\} \tag{2.20}$$

where

$$\overset{N-1}{\psi}(\overset{N}{o}{}^{A}; \overset{N}{x}) = \overset{N}{o}{}^{A}\overset{N-1}{\Psi}{}_{A}(\overset{N}{x}), \qquad \underset{N-1}{\chi}(\underset{N}{\bar{o}}{}^{A'}; \overset{N}{x}) = \underset{N}{\bar{o}}{}^{A'}\underset{N-1}{X}{}^{B'}(\overset{N}{x})\varepsilon_{B'A'} \tag{2.21}$$

are complex scalar functions on C^{+}_{N-1} at $\overset{N}{x}{}^{AA'} \in V(\zeta_{N+1})$. For successive mass-scattering processes, we thus have the infinite NID set

$$\text{IMSNIDS} = \{\overset{1}{\psi}(\overset{2}{o}{}^{A}; \overset{2}{x}), \underset{1}{\chi}(\underset{2}{\bar{o}}{}^{A'}; \overset{2}{x}), \overset{2}{\psi}(\overset{3}{o}{}^{A}; \overset{3}{x}), \underset{2}{\chi}(\underset{3}{\bar{o}}{}^{A'}; \overset{3}{x}), \ldots\} \tag{2.22}$$

In order to rewrite the DFDE in a symbolic form, we now introduce the conjugate spin scalar operators at $\overset{N}{x}{}^{AA'}$

$$\underset{N}{\hat{\pi}}{}_{1/2-} = (\overset{N}{r}/\overset{N}{z})\{\overset{N}{\mathbb{D}} - 2\overset{N-1}{\rho}(\overset{N}{x})\} = (\overset{N}{r}/\overset{N}{z})\overset{N}{\mathfrak{p}}_{\mathrm{L}} \tag{2.23}$$

$$\underset{N}{\hat{\pi}}{}_{1/2+} = (\overset{N}{r}/\underset{N}{\bar{z}})\{\overset{N}{\mathbb{D}} - 2\underset{N-1}{R}(\overset{N}{x})\} = (\overset{N}{r}/\underset{N}{\bar{z}})\underset{N}{\mathfrak{p}}_{\mathrm{R}} \tag{2.24}$$

In these expressions, $\overset{N}{r} \in E(\zeta_{N+1})$ and $\overset{N}{\mathbb{D}}$ is the differentiation operator in the direction of γ_N at $\overset{N}{x}{}^{AA'}$. The functions $\overset{N-1}{\rho}(\overset{N}{x})$ and $\underset{N-1}{R}(\overset{N}{x})$ are real and measure the convergence of the generators of C^{+}_{N-1} at $\overset{N}{x}{}^{AA'}$ (Penrose, 1980). Their defining expression is

$$\overset{N-1}{\rho}(\overset{N}{x}) = (\overset{N}{z})^{-1}\overset{N}{o}{}_{A}\overset{N+1}{o}{}^{C}\underset{N}{\bar{o}}{}^{C'}(\partial/\partial\overset{N}{x}{}^{CC'})\overset{N}{o}{}^{A} = \underset{N-1}{R}(\overset{N}{x}) \tag{2.25}$$

The operators $\overset{N}{\mathfrak{p}}_{\mathrm{L}}$ and $\underset{N}{\mathfrak{p}}_{\mathrm{R}}$ are the usual real forms of the compacted spin-coefficient derivative operator at $\overset{N}{x}{}^{AA'}$ (Penrose and Rindler, 1984). In fact, for the general case of spin s, the $\hat{\pi}$ operators are given in Cardoso (1990*a*). Letting (2.23) and (2.24) act appropriately on the elements of (2.20), we get for the $(N-1)$th process

$$\hat{\pi}\text{-MSNIDS} = \{\underset{N}{\hat{\pi}}{}_{1/2-}\overset{N-1}{\psi}(\overset{N}{o}{}^{A}; \overset{N}{x}), \underset{N}{\hat{\pi}}{}_{1/2+}\underset{N-1}{\chi}(\underset{N}{\bar{o}}{}^{A'}; \overset{N}{x})\} \tag{2.26}$$

The general pattern of the $\hat{\pi}$-NID for successive mass-scattering processes now becomes clear. We have the infinite $\hat{\pi}$-NID set

$$\hat{\pi}\text{-IMSNIDS} = \{\underset{2}{\hat{\pi}}_{1/2-}\overset{1}{\psi}(\overset{2}{o}^{A};\overset{2}{x}),\ \underset{2}{\hat{\pi}}_{1/2+}\underset{1}{\chi}(\underset{2}{\bar{o}}^{A'};\overset{2}{x}),\ \underset{3}{\hat{\pi}}_{1/2-}\overset{2}{\psi}(\overset{3}{o}^{A};\overset{3}{x}),\ \underset{3}{\hat{\pi}}_{1/2+}\underset{2}{\chi}(\underset{3}{\bar{o}}^{A'};\overset{3}{x}),\ldots\} \tag{2.27}$$

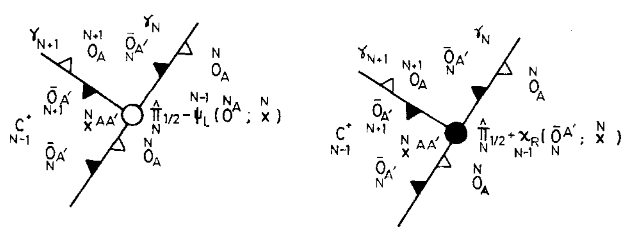
We shall see later (Section 3.2) that the explicit expressions for all the elements of (2.27) can be given in terms of the elements of the following NID set:

$$\hat{\pi}\text{-NIDS} = \{\underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}^{A};\overset{1}{x}),\ \underset{1}{\hat{\pi}}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}^{A'};\overset{1}{x})\} \tag{2.28}$$

where the $\hat{\pi}$ operators are now defined on $\mathscr{C}_0^+$ at $\overset{1}{x}{}^{AA'}$. A diagrammatic representation of the functions appearing as the elements of (2.26) is shown in Figure 2. The $\hat{\pi}$-null datum involving $\overset{N-1}{\psi}$ ($\underset{N-1}{\chi}$) has been represented by a white (black) datum spot. These datum spots bear the letters L and R, which stand for left and right, respectively. Such a diagrammatic representation shall be used also in Section 3. We will effectively make use of the corresponding terminology in Section 4.

As mentioned before, each of the elements of (2.14) propagates for a while along the edges of the relevant MSNZZs. Thus, by splitting

$$\overset{N-2}{\psi}{}_{A}(\overset{N}{x}) \quad \text{and} \quad \underset{N-2}{\chi}{}^{A'}(\overset{N}{x})$$



(a) (b)

Fig. 2. The $\hat{\pi}$-NID for the $(N-1)$th mass-scattering process are represented by datum spots centered at the element $\overset{N}{x}{}^{AA'}$ of the vertex set of an MSNZZ. (a) The ψ datum is denoted by a white datum spot; (b) the χ datum is denoted by a black datum spot.

at $\overset{N-1}{y}{}^{AA'}$, we obtain

$$\overset{N-2}{\Psi}{}_A(\overset{N-1}{y}) = \frac{1}{2\pi} \overset{N}{o}_A \overset{N-1}{\Delta}{}_1(\overset{N-1}{y}) \underset{N-1}{\hat{\pi}}{}_{1/2-} \overset{N-2}{\psi}(\overset{N-1}{o}{}^M; \overset{N-1}{x}) \qquad (2.29)$$

and

$$\underset{N-2}{\chi}{}^{A'}(\overset{N-1}{y}) = \frac{1}{2\pi} \underset{N}{\bar{o}}{}^{A'} \overset{N-1}{\Delta}{}_1(\overset{N-1}{y}) \underset{N-1}{\hat{\pi}}{}_{1/2+} \underset{N-2}{\chi}(\underset{N-1}{\bar{o}}{}^{M'}; \overset{N-1}{x}) \qquad (2.30)$$

where $\overset{N-1}{\Delta}{}_1(\overset{N-1}{y})$ is (Penrose and Rindler, 1984) a proper Lorentz scalar distributional field (PLSDF) on V_0^+ (with the origin displaced to $\overset{N-1}{x}{}^{AA'}$) whose support is C_{N-1}^+. Therefore the massive DFDE (2.16) are rewritten as

$$\overset{N-1}{\nabla}{}_{AA'} \overset{N-1}{\Psi}{}_A(\overset{N-1}{y}) = (\mu/2\pi) \underset{N}{\bar{o}}{}^{A'} \overset{N-1}{\Delta}{}_1(\overset{N-1}{y}) \underset{N-1}{\hat{\pi}}{}_{1/2+} \underset{N-2}{\chi}(\underset{N-1}{\bar{o}}{}^{M'}; \overset{N-1}{x}) \qquad (2.31)$$

$$\overset{N-1}{\nabla}{}_{AA'} \underset{N-1}{\chi}{}^{A'}(\overset{N-1}{y}) = (-\mu/2\pi) \overset{N}{o}_A \overset{N-1}{\Delta}{}_1(\overset{N-1}{y}) \underset{N-1}{\hat{\pi}}{}_{1/2-} \overset{N-2}{\psi}(\overset{N-1}{o}{}^M; \overset{N-1}{x}) \qquad (2.32)$$

Notice that these DFDE are invariant under both Lorentz transformations and arbitrary scalings.

3. MASS-SCATTERING INTEGRALS

To obtain the explicit mass-scattering integral formulas, we need first to solve the DFDE exhibited in Section 2.3. We shall be led to explicit NID for the mass-scattering processes. Each of the elementary fields of (2.14) emerges as an SI mass-scattering integral which is taken over an appropriate space of MSNZZs. The entire Dirac fields appear to be given in terms of four series as each of the expressions (2.13) is split up into two series. We will here use the diagrammatic representation of Section 2.3 whereby the $\hat{\pi}$-NID involved in the scattering processes are represented by white and black datum spots bearing L's and R's.

3.1. Solutions of the DFDE

The solutions of the massless free DFDE (2.17) can be obtained at once by putting $N=2$ in (2.29) and (2.30). We thus have

$$\overset{0}{\Psi}{}_A(\overset{1}{y}) = (1/2\pi) \overset{2}{o}_A \overset{1}{\Delta}{}_1(\overset{1}{y}) \underset{1}{\hat{\pi}}{}_{1/2-} \overset{0}{\psi}(\overset{1}{o}{}^M; \overset{1}{x}) \qquad (3.1)$$

and

$$\underset{0}{\chi}{}^{A'}(\overset{1}{y}) = (1/2\pi) \underset{2}{\bar{o}}{}^{A'} \overset{1}{\Delta}{}_1(\overset{1}{y}) \underset{1}{\hat{\pi}}{}_{1/2+} \underset{0}{\chi}(\underset{1}{\bar{o}}{}^{M'}; \overset{1}{x}) \qquad (3.2)$$

These are the expressions arising from the usual splitting of the Kirchhoff–D'Adhemar–Penrose (KAP) integrals in the case of spin ±1/2 (Penrose and Rindler, 1984; Cardoso, 1990*b*). To the massive DFD equation (2.31), we seek a distributional solution of the form

$$\overset{N-1}{\Psi}{}_A(\overset{N-1}{y}) = \overset{N-1}{\alpha}{}_A(\overset{N-1}{y})\,\overset{N-1}{\Delta}{}_j(\overset{N-1}{y})\,{}_{N-1}\hat{\pi}_{1/2+}\,{}_{N-2}\chi\,({}_{N-1}\bar{o}^{M'};\,\overset{N-1}{x}) \tag{3.3}$$

where $\overset{N-1}{\alpha}{}_A(\overset{N-1}{y})$ is a spinor to be determined, and $\overset{N-1}{\Delta}{}_j(\overset{N-1}{y})$ is a PLSDF defined with respect to $\overset{N-1}{x}{}^{AA'}$. By differentiating out (3.3), using (2.7), (2.8), and the SI distributional relation on V_0^+,

$$\overset{N-1}{\nabla}{}_{AA'}\overset{N-1}{\Delta}{}_j(\overset{N-1}{y}) = \overset{NN}{u o}{}_A{}_{N}\bar{o}_{A'}\,\overset{N-1}{\Delta}{}_{j+1}(\overset{N-1}{y}) - (j/\overset{N-1}{z}\,{}_{N-1}\bar{z}\,\overset{N}{u})\,\overset{N-1}{o}{}_A\,{}_{N-1}\bar{o}_{A'}\,\overset{N-1}{\Delta}{}_j(\overset{N-1}{y}) \tag{3.4}$$

we conclude that $j=0$, and

$$\overset{N-1}{\alpha}{}_A(\overset{N-1}{y}) = -[(\mu/2\pi)/\overset{N-1}{z}\,\overset{N}{u}]\,\overset{N-1}{o}{}_A \tag{3.5}$$

It follows that our SI solution is

$$\overset{N-1}{\Psi}{}_A(\overset{N-1}{y}) = -[(\mu/2\pi)/\overset{N-1}{z}\,\overset{N}{u}]\,\overset{N-1}{o}{}_A\,\overset{N-1}{\Delta}{}_0(\overset{N-1}{y})\,{}_{N-1}\hat{\pi}_{1/2+}\,{}_{N-2}\chi\,({}_{N-1}\bar{o}^{M'};\,\overset{N-1}{x}) \tag{3.6}$$

$\overset{N-1}{\Delta}{}_0(\overset{N-1}{y})$ being (Penrose and Rindler, 1984) a proper Lorentz-invariant step function on V_0^+ defined with respect to $\overset{N-1}{x}{}^{AA'}$. A similar procedure leads to

$$\underset{N-1}{\chi}{}^{A'}(\overset{N-1}{y}) = -[(\mu/2\pi)/{}_{N-1}\bar{z}\,\overset{N}{u}]\,{}_{N-1}\bar{o}^{A'}\,\overset{N-1}{\Delta}{}_0(\overset{N-1}{y})\,{}_{N-1}\hat{\pi}_{1/2-}\,\overset{N-2}{\psi}\,(\overset{N-1}{o}{}^{M};\,\overset{N-1}{x}) \tag{3.7}$$

as the SI solution of (3.2). Since we are concerned here only with forward MSNZZs, we shall henceforth take $\overset{N-1}{\Delta}{}_0(\overset{N-1}{y}) = 1$.

3.2. Explicit NID Expressions for Successive Mass-Scattering Processes

The SI solutions (3.6) and (3.7) give rise to certain symbolic relations involving the elements of (2.27). These relations are easily established by making use of the defining expressions (2.21), (2.23), and (2.24). We thus have

$$\underset{N}{\hat{\pi}}_{1/2-}\overset{N-1}{\psi}(\overset{N}{o}{}^{A};\overset{N}{x})=[(\mu/2\pi)/\overset{NN}{zr}]_{N}\underset{N-1}{\hat{\pi}}_{1/2+}\underset{N-2}{\chi}(\underset{N-1}{\bar{o}}{}^{A'};\overset{N-1}{x}) \tag{3.8}$$

and

$$\underset{N}{\hat{\pi}}_{1/2+}\underset{N}{\chi}(\underset{N}{\bar{o}}{}^{A'};\overset{N}{x})=[(\mu/2\pi)/\underset{N}{\bar{z}}\overset{N}{r}]_{N}\underset{N-1}{\hat{\pi}}_{1/2-}\overset{N-2}{\psi}(\overset{N-1}{o}{}^{A};\overset{N-1}{x}) \tag{3.9}$$

Let us now start with $\underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}{}^{A};\overset{1}{x})$ on $\mathscr{C}_0^+$ at $\overset{1}{x}{}^{AA'}\in V(\lambda_{2K+2})$, where $K\in\mathbb{N}\cup\{0\}$, $\mathbb{N}$ being the set of natural numbers. Using (3.8) and (3.9), we obtain the following expression for the NI datum for the (2Kth-process) mass scattering of the SI field density $\overset{2K}{\Psi}_{A}(\overset{2K}{y})$ on C_{2K}^+ at $\overset{2K+1}{x}{}^{AA'}\in V(\lambda_{2K+2})$:

$$\begin{aligned}&\underset{2K+1}{\hat{\pi}}_{1/2-}\overset{2K}{\psi}(\overset{2K+1}{o}{}^{A};\overset{2K+1}{x})\\&=(\mu/2\pi)^{2K}\underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}{}^{A};\overset{1}{x})\Big/\left[\left(\prod_{n=1}^{K}\overset{2n+1}{z}\right)\left(\prod_{m=1}^{K}\underset{2m}{\bar{z}}\right)\left(\prod_{j=1}^{2K}\overset{j+1}{r}\right)\right]\end{aligned} \tag{3.10}$$

where the explicitly involved affine parameters belong to $E(\lambda_{2K+2})$ whenever K takes on the values 1, 2, 3, ... Similarly, starting with

$$\underset{1}{\hat{\pi}}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x})\quad\text{on}\quad\mathscr{C}_0^+\quad\text{at}\quad\overset{1}{x}{}^{AA'}\in V(\eta_{2K+3})$$

we obtain

$$\begin{aligned}&\underset{2K+2}{\hat{\pi}}_{1/2-}\overset{2K+1}{\psi}(\overset{2K+2}{o}{}^{A};\overset{2K+2}{x})\\&=(\mu/2\pi)^{2K+1}\underset{1}{\hat{\pi}}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x})\Big/\left[\left(\prod_{m=1}^{K+1}\overset{2m}{z}\right)\left(\prod_{n=1}^{K}\underset{2n+1}{\bar{z}}\right)\left(\prod_{j=1}^{2K+1}\overset{j+1}{r}\right)\right]\end{aligned} \tag{3.11}$$

which is the expression for the NI datum for the [(2K+1)th-process] mass scattering of the (SI)

$$\overset{2K+1}{\Psi}_{A}(\overset{2K+1}{y})\quad\text{on}\quad C_{2K+1}^+\quad\text{at}\quad\overset{2K+2}{x}{}^{AA'}\in V(\eta_{2K+3})$$

It is clear that the affine parameters involved in (3.11) all belong to $E(\eta_{2K+3})$.

The results for the mass-scattering processes involving the SI densities

$$\underset{2K}{X}{}^{A'}(\overset{2K}{y})\quad\text{and}\quad\underset{2K+1}{X}{}^{A'}(\overset{2K+1}{y})$$

can be obtained from the previous ones as follows. We first make the replacements

$$\underset{2K+1}{\hat{\pi}}{}_{1/2-}\overset{2K}{\psi}(\overset{2K+1}{o}{}^{A};\overset{2K+1}{x}) \to \underset{2K+1}{\hat{\pi}}{}_{1/2+}\underset{2K}{\chi}(\underset{2K+1}{\bar{o}}{}^{A'};\overset{2K+1}{x})$$

$$\underset{2K+2}{\hat{\pi}}{}_{1/2-}\overset{2K+1}{\psi}(\overset{2K+2}{o}{}^{A};\overset{2K+2}{x}) \to \underset{2K+2}{\hat{\pi}}{}_{1/2+}\underset{2K+1}{\chi}(\underset{2K+2}{\bar{o}}{}^{A'};\overset{2K+2}{x})$$

Next we apply the interchange rule

$$\underset{1}{\hat{\pi}}{}_{1/2-}\overset{0}{\psi}(\overset{1}{o}{}^{A};\overset{1}{x}) \leftrightarrow \underset{1}{\hat{\pi}}{}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x})$$

Finally we take the complex conjugates of the explicitly involved inner products. We thus get

$$\begin{aligned}&\underset{2K+1}{\hat{\pi}}{}_{1/2+}\underset{2K}{\chi}(\underset{2K+1}{\bar{o}}{}^{A'};\overset{2K+1}{x})\\ &\quad=(\mu/2\pi)^{2K}\underset{1}{\hat{\pi}}{}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x})\Big/\left[\left(\prod_{m=1}^{K}\overset{2m}{z}\right)\left(\prod_{n=1}^{K}\underset{2n+1}{\bar{z}}\right)\left(\prod_{j=1}^{2K}\overset{j+1}{r}\right)\right]\end{aligned} \tag{3.12}$$

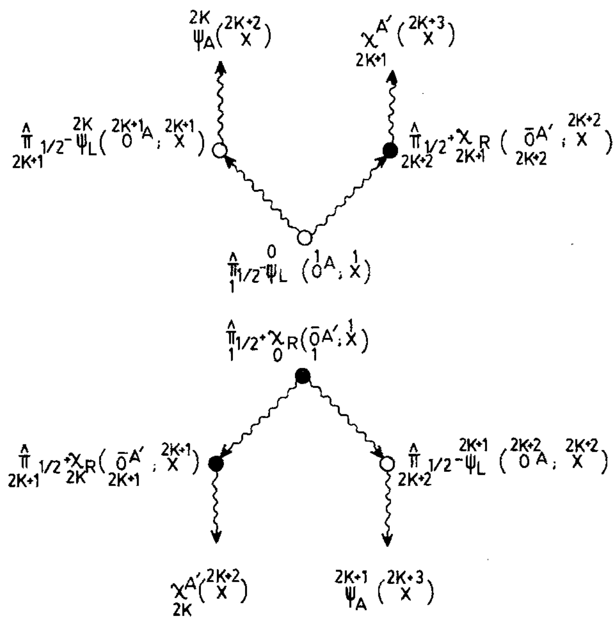


Fig. 3. Scheme showing the generation of the NID for the mass-scattering processes involving the SI distributional field densities in V_0^+.

on C^+_{2K} at $\overset{2K+1}{x}{}^{AA'} \in V(\lambda_{2K+2})$, and

$$\underset{2K+2}{\hat{\pi}}{}^{1/2+} \underset{2K+1}{\chi}\big(\underset{2K+2}{\bar{o}}{}^{A'};\ \overset{2K+2}{x}\big)$$
$$= (\mu/2\pi)^{2K+1} \underset{1}{\hat{\pi}}{}_{1/2-} \overset{0}{\psi}(\overset{1}{o}{}^{A};\ \overset{1}{x}) \Bigg/ \left[\left(\prod_{n=1}^{K} \overset{2n+1}{z}\right)\left(\prod_{m=1}^{K+1} \underset{2m}{\bar{z}}\right)\left(\prod_{j=1}^{2K+1} \overset{j+1}{r}\right)\right]$$

on C^+_{2K+1} at $\overset{2K+2}{x}{}^{AA'} \in V(\eta_{2K+3})$. A schematic representation of the above procedure is shown in Figure 3.

It is now evident that the elements of (2.28) are the NID for the "zeroth-mass-scattering process." These are the NID that enter into a modified SI version of the KAP field integrals in the case of spin $\pm 1/2$ (Cardoso, 1990*a*). The explicit p-NID for successive mass-scattering processes are given in Cardoso (1988).

3.3. The Entire Dirac Fields

For writing down the SI mass-scattering integrals for the elements of the infinite Dirac set (2.14), we need to define a mass-scattering $(3j-1)$-differential form, $j \in \mathbb{N}$. Such a form is an SI $(3j-1)$-volume form $\overset{123...j}{\underset{\sim}{K}}$ on the compact space $\overset{123...j}{\mathbb{K}}$ of MSNZZs whose edge-sets possess $j+1$ edges. The relevant defining expression is

$$\overset{123...j}{\underset{\sim}{K}} = \left(\bigwedge_{m=1}^{j-1} \overset{m}{\underset{\sim}{r}} \wedge \overset{m}{\underset{\sim}{S}}\right) \wedge \overset{i}{\underset{\sim}{K}} \tag{3.14}$$

where $\overset{m}{\underset{\sim}{r}}$ is the SI one-form $d\overset{m}{r}/\overset{m}{r}$ at the element $\overset{m}{x}{}^{AA'}$ of the vertex-set of some $\zeta_{j+1} \in \overset{123...j}{\mathbb{K}}$, with $\overset{m}{r} \in E(\zeta_{j+1})$; $\overset{m}{\underset{\sim}{S}}$ is the SI two-form of surface area provided by the (spacelike) intersection of the (past) backward null cone C^-_{m+1} of $\overset{m+1}{x}{}^{AA'} \in V(\zeta_{j+1})$ with the (future) forward null cone C^+_{m-1} of $\overset{m-1}{x}{}^{AA'} \in V(\zeta_{j+1})$, and $\overset{j}{\underset{\sim}{K}}$ is an SI two-form on the (two-dimensional) space of pairs of adjacent edges $\overset{j}{r}$ and $\overset{j+1}{r}$ of $E(\zeta_{j+1})$ incident at $\overset{j}{x}{}^{AA'} \in V(\zeta_{j+1})$, which is given by

$$\overset{j}{\underset{\sim}{K}} = \overset{j}{\underset{\sim}{S}} / (\overset{i}{z} \underset{j}{\bar{z}} \overset{j}{r} \overset{j+1}{r}) \tag{3.15}$$

We are now in a position to introduce the SI mass-scattering formulas. We have

$$\overset{2K}{\psi}_A(\overset{2K+2}{x}) = \frac{\mu^{2K}}{(2\pi)^{2K+1}} \int_{\underset{\mathbb{K}}{123\ldots 2K+1}} \overset{2K+2}{o}{}_A \times \frac{\underset{1}{\hat{\pi}}_{1/2-} \psi(\overset{0}{o}{}^{\overset{1}{M}}; \overset{1}{x})^{\underset{\sim}{K}^{123\ldots 2K+1}}}{\left(\prod_{n=1}^{K} \overset{2n+1}{z}\right)\left(\prod_{m=1}^{K} \underset{2m}{\bar{z}}\right)\left(\prod_{p=1}^{2K} \overset{p+1}{r}\right)} \tag{3.16}$$

$$\overset{2K+1}{\psi}{}_A(\overset{2K+3}{x}) = \frac{\mu^{2K+1}}{(2\pi)^{2K+2}} \int_{\underset{\mathbb{K}}{123\ldots 2K+2}} \overset{2K+3}{o}{}_A \times \frac{\underset{1}{\hat{\pi}}_{1/2+} \chi(\underset{0}{\bar{o}}{}^{M'}; \underset{1}{x})^{\underset{\sim}{K}^{123\ldots 2K+2}}}{\left(\prod_{m=1}^{K+1} \overset{2m}{z}\right)\left(\prod_{n=1}^{K} \underset{2n+1}{\bar{z}}\right)\left(\prod_{p=1}^{2K+1} \overset{p+1}{r}\right)} \tag{3.17}$$

for the unprimed elements of (2.14), and

$$\underset{2K}{\chi}{}^{A'}(\overset{2K+2}{x}) = \frac{\mu^{2K}}{(2\pi)^{2K+1}} \int_{\underset{\mathbb{K}}{123\ldots 2K+1}} \underset{2K+2}{\bar{o}}{}^{A'} \times \frac{\underset{1}{\hat{\pi}}_{1/2+} \chi(\underset{0}{\bar{o}}{}^{M'}; \underset{1}{x})^{\underset{\sim}{K}^{123\ldots 2K+1}}}{\left(\prod_{m=1}^{K} \overset{2m}{z}\right)\left(\prod_{n=1}^{K} \underset{2n+1}{\bar{z}}\right)\left(\prod_{p=1}^{2K} \overset{p+1}{r}\right)} \tag{3.18}$$

$$\underset{2K+1}{\chi}{}^{A'}(\overset{2K+3}{x}) = \frac{\mu^{2K+1}}{(2\pi)^{2K+2}} \int_{\underset{\mathbb{K}}{123\ldots 2K+2}} \underset{2K+3}{\bar{o}}{}^{A'} \times \frac{\underset{1}{\hat{\pi}}_{1/2-} \psi(\overset{0}{o}{}^{\overset{1}{M}}; \overset{1}{x})^{\underset{\sim}{K}^{123\ldots 2K+2}}}{\left(\prod_{n=1}^{K} \overset{2n+1}{z}\right)\left(\prod_{m=1}^{K+1} \underset{2m}{\bar{z}}\right)\left(\prod_{p=1}^{2K+1} \overset{p+1}{r}\right)} \tag{3.19}$$

for the primed elements. Each of the above mass-scattering integrals takes into account all the contributions coming from the spotted vertices of the appropriate MSNZZs (see Figure 2). Additionally, it is evident that these integrals carry explicitly the NID expressions given previously. These

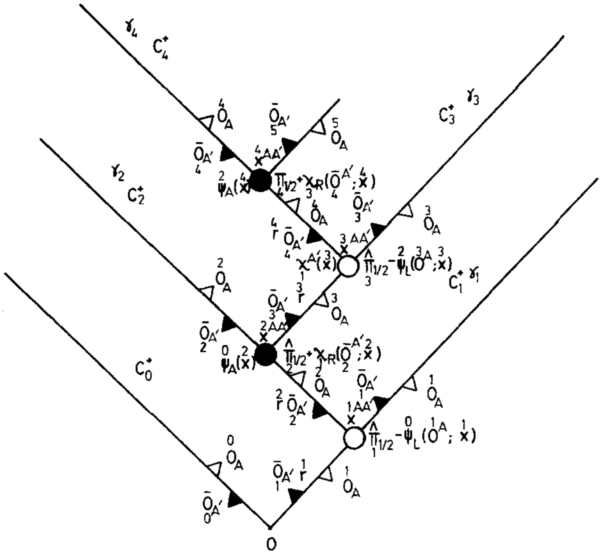


Fig. 4. Diagram showing successive mass-scattering processes of the SI distributional field densities on V_0^+. The NID for the processes are generated by $\underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}^{A}; \overset{1}{x})$ on $\mathscr{C}_0^+$.

features are illustrated in Figs. 4 and 5. Particularly, the integral expressions for the massless free elementary fields are written out explicitly as

$$\overset{0}{\psi}_A(\overset{2}{x})=\frac{1}{2\pi}\int_{\overset{1}{\mathbb{K}}} \overset{2}{o}_A \underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}^{M}; \overset{1}{x})\overset{1}{\underset{\sim}{K}} \tag{3.20}$$

and

$$\underset{0}{\chi}^{A'}(\overset{2}{x})=\frac{1}{2\pi}\int_{\overset{1}{\mathbb{K}}} \underset{2}{\bar{o}}^{A'} \underset{1}{\hat{\pi}}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}^{M'}; \overset{1}{x})\overset{1}{\underset{\sim}{K}} \tag{3.21}$$

These are the SI version of the KAP field integrals for the case of spin $\pm 1/2$. The SI expressions for the general case of spin s are given in Cardoso (1990*a*).

Simple formal mass-scattering integral expressions can be achieved by defining suitable SI field densities on C_{2K}^+ and C_{2K+1}^+, such that (see Figure 6)

$$\overset{2K}{\psi}_A(\overset{2K+2}{x})=\frac{1}{2\pi}\int_{\overset{123\ldots 2K+1}{\mathbb{K}}} \overset{2K}{\Psi}_A(\overset{2K+1}{x})\overset{123\ldots 2K+1}{\underset{\sim}{K}} \tag{3.22}$$

$$\overset{2K+1}{\psi}_A(\overset{2K+3}{x})=\frac{1}{2\pi}\int_{\overset{123\ldots 2K+2}{\mathbb{K}}} \overset{2K+1}{\Psi}_A(\overset{2K+2}{x})\overset{123\ldots 2K+2}{\underset{\sim}{K}} \tag{3.23}$$

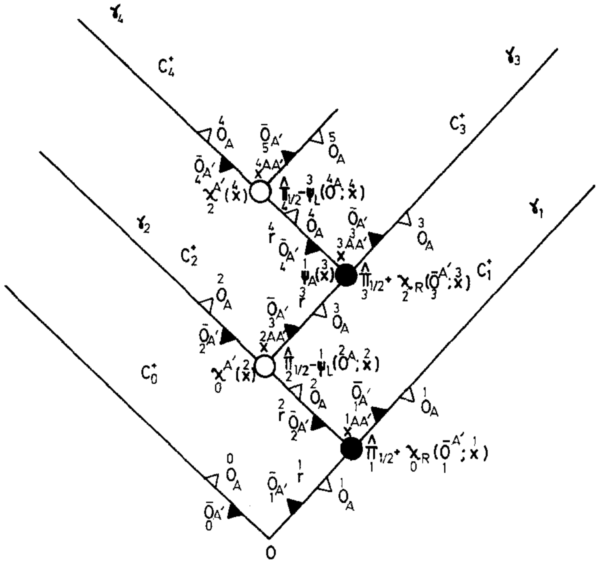


Fig. 5. Diagram showing successive mass-scattering processes of the SI distributional field densities on V_0^+. The NID for the processes are generated by $\underset{1}{\hat{\pi}}_{1/2+}\underset{0}{\chi}(\bar{o}^{A'};\overset{1}{\underset{1}{x}})$ on $\mathscr{C}_0^+$.

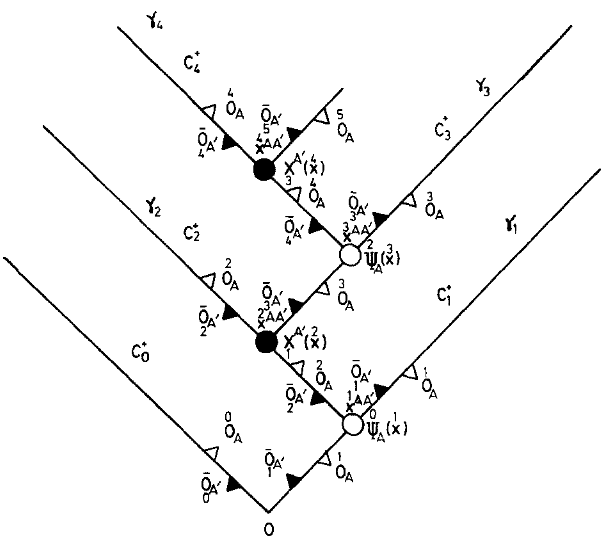


Fig. 6. The SI field densities on V_0^+ generated by $\underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}^{A};\overset{1}{x})$ on $\mathscr{C}_0^+$. These densities define simple formal mass-scattering integrals for the elementary fields $\underset{2K}{\psi}_A(x)$ and $\underset{2K+1}{\chi}{}^{A'}(x)$.

and

$$\underset{2K}{\chi}^{A'}(\overset{2K+2}{x}) = \frac{1}{2\pi}\int_{\overset{123\ldots 2K+1}{\mathbb{K}}} \underset{2K}{X}^{A'}(\overset{2K+1}{x})\,\overset{123\ldots 2K+1}{\underset{\sim}{K}} \tag{3.24}$$

$$\underset{2K+1}{\chi}^{A'}(\overset{2K+3}{x}) = \frac{1}{2\pi}\int_{\overset{123\ldots 2K+2}{\mathbb{K}}} \underset{2K+1}{X}^{A'}(\overset{2K+2}{x})\,\overset{123\ldots 2K+2}{\underset{\sim}{K}} \tag{3.25}$$

Hence the entire fields (2.13) are reexpressed as

$$\psi_A(x) = \sum_{K=0}^{\infty} \overset{2K}{\psi}_A(\underset{\lambda}{\overset{2K+2}{x}}) + \sum_{K=0}^{\infty} \overset{2K+1}{\psi}_A(\underset{\eta}{\overset{2K+3}{x}}) \tag{3.26}$$

$$\chi^{A'}(x) = \sum_{K=0}^{\infty} \underset{2K}{\chi}^{A'}(\underset{\lambda}{\overset{2K+2}{x}}) + \sum_{K=0}^{\infty} \underset{2K+1}{\chi}^{A'}(\underset{\eta}{\overset{2K+3}{x}}) \tag{3.27}$$

A schematic diagram showing this recovery is given in Figure 7.

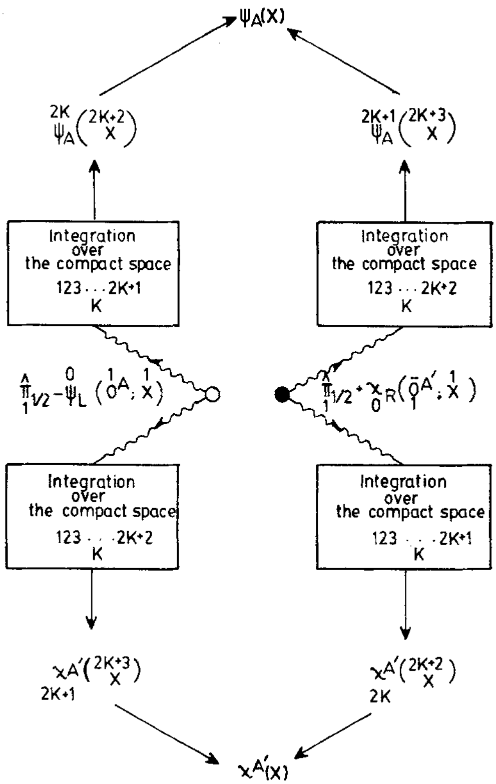


Fig. 7. Schematic representation of the recovery of the entire Dirac field set on V_0^+.

The diagrams illustrating our mass-scattering scheme, which have been exhibited above, can enable one to draw all the other relevant scattering diagrams. It is worth remarking, in particular, that the diagrams shown in Figures 4 and 5 can be obtained from one another by interchanging the datum spots appropriately (see Section 4).

4. GRAPHICAL DESCRIPTION OF THE MASS-SCATTERING PROCESSES

Basically, we shall now show how the information carried by the generalized mass-scattering integral expressions exhibited in the foregoing section can be extracted from certain simple graphs. Such graphs will henceforth be designated as mass-scattering graphs (MSGs). The unprimed and primed elementary fields shall be called left-handed and right-handed fields, respectively. For the sake of convenience, we will relabel the elements of the edge-sets of the scattering diagrams which appear to be of relevance to us.

4.1. Mass-Scattering Graphs

An MSG is a (simple) connected oriented colored graph σ_{N+2} ($N \geq 0$) (Busacker and Saaty, 1965; Nakanishi, 1971) whose vertex-set

$$V(\sigma_{N+2}) = \{\overset{0}{v}, \overset{1}{v}, \overset{2}{v}, \overset{3}{v}, \ldots, \overset{N+2}{v}\} \tag{4.1}$$

contains $N+3$ elements such that $\overset{n+1}{v}$ is "forwardly" separated from $\overset{n}{v}$, $n = 0, 1, 2, \ldots, N+1$. Indeed, this "forwardness" is what defines the orientation of σ_{N+2}. The edge-set of σ_{N+2} is the set

$$E(\sigma_{N+2}) = \{\overset{1}{a}, \overset{2}{a}, \overset{3}{a}, \ldots, \overset{N+2}{a}\} \tag{4.2}$$

of $N+2$ edges, where $\overset{n+1}{a}$ connects the vertices $\overset{n}{v}$ and $\overset{n+1}{v}$. We now introduce a suitable one-to-one correspondence between the (vertex-sets) edge-sets of MSNZZs and the (vertex-sets) edge-sets of MSGs. We have

$$\vartheta: \quad V(\zeta_{N+2}) \to V(\sigma_{N+2}) \tag{4.3a}$$

$$\varepsilon: \quad E(\zeta_{N+2}) \to E(\sigma_{N+2}) \tag{4.3b}$$

which establish the relationship between $(\overset{h}{x}{}^{AA'})\overset{n+1}{r}$ and $(\overset{h}{v})\overset{n+1}{a}$, $h = 0, 1, 2, \ldots, N+2$. Both the starting-vertex $(\overset{0}{v})$ and the end-vertex $(\overset{N+2}{v})$ of σ_{N+2} carry no color, while each of the "internal" vertices carries either the color white or the color black. It becomes evident that the number of colored

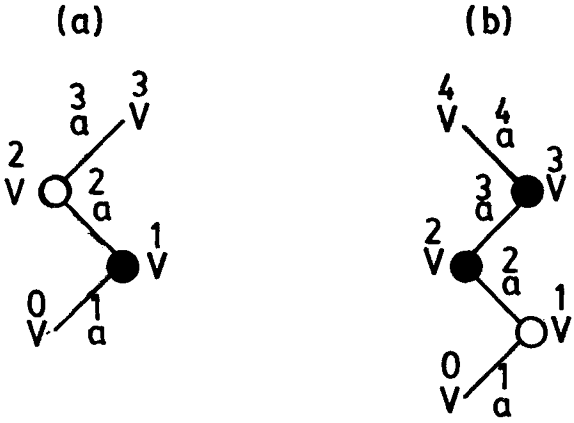


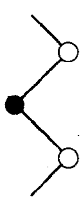
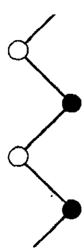
Fig. 8. Two zigzag-like patterns: (a) An MSG; (b) a zigzag which is not an MSG.

vertices of σ_{N+2} is equal to $N+1$. For $N>0$, any internal edge of $E(\sigma_{N+2})$ joins only vertices carrying different colors. Each white (black) vertex is associated with the appropriate NI datum for the scattering of a left-handed (right-handed) field. For example, we can have a graph as shown in Figure 8a, but not as shown in Figure 8b.

Notice that what appears to be of importance at this stage is the colored-vertex configuration together with the number of involved edges. For convenience, only zigzag-like MSGs are considered here. From now on we shall for simplicity drop the v's and a's from the scattering graphs.

4.2. Graphical Representation of the Mass-Scattering Integrals

Consider the left-handed outgoing field $\overset{N}{\psi}_A(x)$, which is involved in the Nth mass-scattering process. This process is graphically described by a σ_{N+2}. The vertex $\overset{1}{v}$ of this graph carries a white or black color, depending upon whether $N+2$ is even or odd. For example,

$\overset{2}{\psi}_A(x) =$  , $\overset{3}{\psi}_A(x) =$  (4.4)

The case $N=0$ corresponds now to the "zeroth process," which involves the massless-free elementary contribution. We thus have

$$\overset{0}{\psi}_A(x)= \tag{4.5}$$

For $N>0$ the $\hat{\pi}$-NI datum for the process involves explicitly the product of N affine parameters and N appropriate spin-inner products in the denominator of the integrand of the relevant mass-scattering integral. The involved parameters are set as

$$\mathrm{LAP}_N=\{\overset{2}{r}, \overset{3}{r}, \ldots, \overset{N+1}{r}\}\subset E(\zeta_{N+2}) \tag{4.6}$$

while the involved inner products are given by

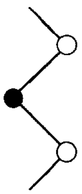
$$\mathrm{ELIP}_N=\{\underset{2}{\bar{z}}, \overset{3}{z}, \underset{4}{\bar{z}}, \overset{5}{z}, \ldots, \underset{N}{\bar{z}}, \overset{N+1}{z}\} \tag{4.7a}$$

in the even-order case, and by

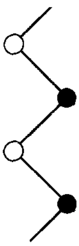
$$\mathrm{OLIP}_N=\{\overset{2}{z}, \underset{3}{\bar{z}}, \overset{4}{z}, \underset{5}{\bar{z}}, \ldots, \underset{N}{\bar{z}}, \overset{N+1}{z}\} \tag{4.7b}$$

in the odd-order case. We observe that the elements of (4.6) correspond to the internal edges of σ_{N+2}. For (4.7a), the inner product $\overset{1}{z}$ is (suitably) implicitly carried by $(\mu/2\pi)^N \underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}{}^A; \overset{1}{x})$, while for (4.7b) the $\hat{\pi}$-NI datum $(\mu/2\pi)^N \underset{1}{\hat{\pi}}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}{}^{A'}; \overset{1}{x})$ already involves $\underset{1}{\bar{z}}$ adequately. Actually, each spin-inner product entering into the relevant mass-scattering integral is defined at a vertex of MSNZZs which corresponds to a colored vertex of MSGs. This inner product is barred or unbarred according as the corresponding colored vertex is black or white. It follows that, in the even-order (odd-order) case, $V(\sigma_{N+2})$ possesses $N/2+1$ [resp. $(N+1)/2$] white vertices and $N/2$ [resp. $(N+1)/2$] black vertices. In either case, the numerator of the integrand of the scattering integral involves the "outgoing" spinor $\overset{N+2}{o}{}_A$ together with a $(3N+2)$-differential form $\overset{123\ldots N+1}{\underset{\sim}{K}}$ on $\overset{123\ldots N+1}{\mathbb{K}}$, multiplied by the appropriate $\hat{\pi}$-NI datum at $\overset{1}{x}{}^{AA'}$. For example, for the graphs

(4.4), we have



$$= \left(\frac{\mu}{2\pi}\right)^2 \frac{1}{2\pi} \int_{\overset{123}{\mathbb{K}}} \overset{4}{o}_A \frac{\underset{1}{\hat{\pi}}_{1/2-}\overset{0}{\psi}(\overset{1}{o}^A;\overset{1}{x})}{(\underset{2}{\overset{3}{\bar{z}}}z)(\overset{2}{r}\overset{3}{r})} \overset{123}{\underset{\sim}{K}} \tag{4.8}$$



$$= \left(\frac{\mu}{2\pi}\right)^3 \frac{1}{2\pi} \int_{\overset{1234}{\mathbb{K}}} \overset{5}{o}_A \frac{\underset{1}{\hat{\pi}}_{1/2+}\underset{0}{\chi}(\underset{1}{\bar{o}}^{A'};\overset{1}{x})}{(\overset{2}{z}\underset{3}{\bar{z}}\overset{4}{z})(\overset{2}{r}\overset{3}{r}\overset{4}{r})} \overset{1234}{\underset{\sim}{K}} \tag{4.9}$$

The MSG that describes the scattering of the right-handed elementary $\underset{N}{\chi}^{A'}(x)$ can be obtained from the one for $\underset{N}{\psi}_A(x)$ by interchanging white and black vertices. Thus, the colored-vertex configuration of the corresponding graph appears the other way about, its edge-structure being associated with (see Figures 4 and 5)

$$\mathrm{RAP}_N = \mathrm{LAP}_N \tag{4.10}$$

It follows that, in the massive case, the relevant inner products arising here are obtained from the elements of (4.7a) and (4.7b) by taking a complex conjugation. The rules for this case are essentially the same as the ones given before, but the scattering integral now involves the "outgoing" spinor $\underset{N+2}{\bar{o}}^{A'}$. We are thus led to the colored-vertex structures

$$(\overset{1}{\circ}\ \underset{2}{\bullet}\ \overset{3}{\circ}\ \underset{4}{\bullet}\ \overset{5}{\circ}\ \cdots\ \underset{N}{\bullet}\ \overset{N+1}{\circ}),\quad (\underset{1}{\bullet}\ \overset{2}{\circ}\ \underset{3}{\bullet}\ \overset{4}{\circ}\ \underset{5}{\bullet}\ \cdots\ \underset{N}{\bullet}\ \overset{N+1}{\circ}) \tag{4.11}$$

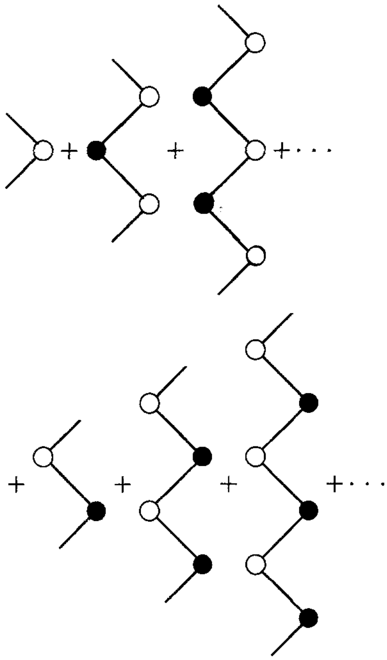
for the left-handed fields, and

$$(\underset{1}{\bullet}\ \overset{2}{\circ}\ \underset{3}{\bullet}\ \overset{4}{\circ}\ \underset{5}{\bullet}\ \cdots\ \overset{N}{\circ}\ \underset{N+1}{\bullet}),\quad (\overset{1}{\circ}\ \underset{2}{\bullet}\ \overset{3}{\circ}\ \underset{4}{\bullet}\ \overset{5}{\circ}\ \cdots\ \overset{N}{\circ}\ \underset{N+1}{\bullet}) \tag{4.12}$$

for the right-handed ones. Evidently, the numbers carried by these structures refer to the labels of the colored vertices of the scattering graphs that represent the pertinent elementary fields. In particular, it should be noticed that the interchange rule of Section 3.2 has been automatically incorporated into this graphical scheme.

According to the above rules, the entire Dirac fields are recovered graphically as

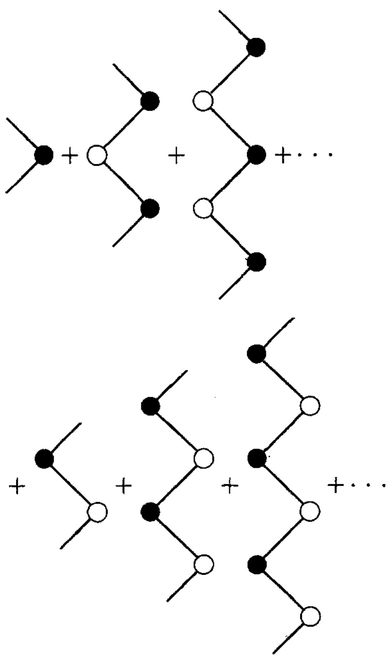
$$\psi_A(x) =$$



(4.13)

and

$$\chi^{A'}(x) =$$



(4.14)

5. CONCLUDING REMARKS AND OUTLOOK

We presented a complete null description of the mass scattering of Dirac fields in $\mathbb{RM}$. One of the most important features of our scheme rests upon the fact that the information on the NID for mass-scattering processes is totally contained in suitably contracted derivatives of spinors entering into FSBSs. Here, the $\hat{\pi}$-spin scalar operators are regarded as being more natural than the $\mathfrak{p}$-operators, insofar as the former are involved in the expressions for the SI field densities. These densities are specified at points lying on forward null cones of appropriate elements of vertex-sets of MSNZZs. It should be emphasized that the particularly simple form of the scattering integrals carrying explicitly such densities is due to the choice of $\mathscr{C}_0^+$ as the NID hypersurface for all the elementary fields. At every order, the number defining the order of either field whose SI mass-scattering integral involves the $(3j-1)$-form $\overset{123\ldots j}{K}$ is equal to $j-1$. The point at which the field is evaluated is the end-vertex of some MSNZZ that possesses $j+1$ edges.

A noteworthy feature of the solutions of our DFDE is the fact that they involve a neutrino field at any order. The usual splitting of the KAP integral expressions for massless free fields thus arose once again here. In fact, as regards the PLSDFs, the mass-scattering integral for a massive contribution of order $j-1$ involves only the product of step-functions $\prod_{n=1}^{j-1} \overset{n}{\Delta}_0(\overset{n}{y})$, which is defined with respect to suitable vertices of the relevant MSNZZs. This was taken equal to 1 here, since our zigzags are forward null graphs in $\mathbb{RM}$. Nevertheless, as the DFDE of Section 2 stand thereupon, the corresponding higher-order contributions can be looked upon as massive pieces which are propagated in V_0^+ by the field equations. Indeed, what arises when the elementary contributions are broken down into distributional fields involving $\Delta_1(x)$ is that they behave themselves as massless pieces which propagate along the edges of MSNZZs. Under these circumstances, in effect, they appear to play the role of sources for the contributions associated with the densities that are involved in the left-hand sides of the DFDE. With respect to the former interpretation, it can be stated that what propagates along the null geodesics containing the edges of the scattering diagrams is the information carried by the elementary fields.

We have assumed from the outset that the series (2.13) giving the entire elements of the Dirac pair are convergent on V_0^+. This situation will probably be discussed elsewhere. The regular behavior of the NID (2.18) at the vertex of $\mathscr{C}_0^+$ is extensively discussed in Penrose and Rindler (1984), and effectively taken into account here. Actually, it provided us with a manifestly finite mass-scattering integral for each of the elementary fields. It may well be said that the relevance of the graphical description given in Section 4 stems from the fact that, once a set of rules for the scattering graphs is at our disposal, we can obtain at once the integral expression for any elementary contribution of either handedness without performing any explicit calculation.

The generalized mass-scattering formulas exhibited in this paper fit neatly with the twistor formalism (see, for instance, Cardoso, 1990*c*). In connection with this fact, it seems to be worthwhile to set up a framework, within the NID approach, which might enable us to deal with quantized fields. It is believed that this procedure would bring new insights into the theory of twistors.

ACKNOWLEDGMENTS

I am deeply indebted to Prof. Roger Penrose for introducing me to the NID techniques. I am grateful to Dr. Asghar Qadir for useful suggestions. I wish to acknowledge the World Laboratory for supporting this work financially.

REFERENCES

Busacker, R. G., and Saaty, T. L. (1965). *Finite Graphs and Networks: An Introduction with Applications*, McGraw-Hill, New York.

Cardoso, J. G. (1988). M. Sc. thesis, University of Oxford.

Cardoso, J. G. (1990*a*). New twistorial integral formulae for massless free fields of arbitrary spin, Preprint, Quaid-i-Azam University.

Cardoso, J. G. (1990*b*). Bessel function expressions for Dirac fields, Preprint, Quaid-i-Azam University.

Cardoso, J. G. (1990*c*). Twistorial transcription of the mass-scattering of Dirac fields, Preprint, Quaid-i-Azam University.

Nakanishi, N. (1971). *Graph Theory and Feynman Integrals*, Gordon and Breach, New York.

Newman, E. T., and Penrose, R. (1968). *Proceedings of the Royal Society of London A*, **305**, 175.

Penrose, R. (1963). In *Relativity, Groups and Topology: The 1963 Les Houches Lectures*, B. S. DeWitt and C. M. DeWitt, eds., New York.

Penrose, R. (1980). *General Relativity and Gravitation*, **12**, 225.

Penrose, R., and Rindler, W. (1984). *Spinors and Space-Time*, Vol. 1, Cambridge University Press, Cambridge.